ORC 66-32
October 1966

THE k SHORTEST ROUTES AND THE k SHORTEST CHAINS IN A GRAPH

by

Michel Sakarovitch

OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF CALIFORNIA - BERKELEY

THE k SHORTEST ROUTES AND THE k SHORTEST CHAINS IN A GRAPH

by

Michel Sakarovitch
Operations Research Center
University of California, Berkeley

October 1966 ORC 66-32

This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83), the National Science Foundation under Grant GP-4593, and the National Bureau of Standards, Contract CST-136 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

THE k SHORTEST ROUTES AND THE k SHORTEST CHAINS IN A GRAPH

by

Michel Sakarovitch

1. Introduction

In a given graph $G = (N,A)$ where N represents the set of nodes and A the set of directed arcs, a length $a(x,y)$ is associated to every arc $(x,y) \in A$. A route from x_o to x_n is a sequence of nodes and arcs

$$P = [x_o,(x_o,x_1),x_1,(x_1,x_2) ,\dots, x_{n-1},(x_{n-1},x_n),x_n]$$

such that

$$x_i \in N \qquad i = 0,\dots,n$$

$$(x_{i-1},x_i) \in A \qquad i = 1,\dots,n$$

A route P will also be considered as a partial subgraph (see [2]) of G containing nodes $x_o,\dots,x_n$ and arcs $(x_o,x_1) ,\dots, (x_{n-1},x_n)$. The length of a route P is:

$$\ell(P) = \sum_{i=1}^{n} a(x_{i-1},x_i)$$

A chain is a route in which no node is repeated. A cycle is a route in which no node is repeated except that $x_o = x_n$. O and D will be two particular nodes. $P_q(P'_q)$ will denote the q^{th} shortest route (chain) from O to D and:

$$G_q = (N_q,A_q) = \bigcup_{j=1}^{q} P_j$$

will be the partial subgraph of G defined by:

$$x \in N_q \qquad \text{iff} \quad x \in P_j \quad \text{for some} \quad j \leq q$$

$$(x,y) \in A_q \qquad \text{iff} \quad (x,y) \in P_j \quad \text{for some} \quad j \leq q .$$

$$\bar{A}_q = A - A_q$$

In a similar way G'_q, A'_q, $\bar{A}'_q$ and N'_q are defined. k_q is the number of routes in G_q which we need to generate as $P^q_{q+1}, \ldots, P^q_{q+k_q}$ i.e. k_q is the min of $(k - q)$ and (max number of routes in $G_q - q$) .

Our central problem will be the construction of the k shortest routes (k shortest chains) from 0 to D . In this form, the problem is slightly more general than the one dealt with in [1], [6] and [3] and reviewed in [7] in two respects:

- Negative lengths are allowed provided that no: "negative cycle" (a cycle along which $\sum a(x,y) < 0$) exist.
- Attention is not restricted to chains.

Moreover, this paper is based on the following simple idea:

At any stage of the algorithm, suppose we know the q $(q < k)$ shortest routes from 0 to D and $P^q_{q+1}, \ldots, P^q_{q+k_q}$ which are the $(q+1)^{st}, \ldots, (q+k_q)^{th}$ shortest routes in G_q ordered according to increasing lengths. $\bar{P}^q_{q+1}$ is a shortest route from 0 to D which has at least one arc in $\bar{A}_q$. The $(q+1)^{st}$ shortest route is obtained by:

$$P_{q+1} = \begin{cases} P^q_{q+1} & \text{if} \quad \ell(P^q_{q+1}) \leq \ell(\bar{P}^q_{q+1}) \\ \bar{P}^q_{q+1} & \text{if} \quad \ell(\bar{P}^q_{q+1}) < \ell(P^q_{q+1}) \end{cases}$$

G_q and the list P^q_{q+i} is updated and the process begins again.

The algorithm is initiated by finding the shortest routes from 0 to every node using an algorithm such as the one in [4] if all lengths are nonnegative and the one in [5] if there are negative lengths.

In the next section an algorithm to construct the k shortest routes from 0 to D is described and it is justified in Section 3. Since the problem of finding the k shortest chains is of interest in many applications, a variant of the algorithm presented in Section 2 which exhibits the k shortest chains is described

in Section 4. Attention is drawn to the fact that minor modifications of these two algorithms allow to find the routes (chains) shorter than a given length. If capacities were to be considered, an immediate application of this last problem could be: what quantity can be shipped from 0 to D if we demand that the maximum time of travel be less than or equal to a certain given number.

2. Algorithm For Finding The k Shortest Routes

Step 0. - Find all the shortest routes from 0 to D : $P_1,\ldots,P_q$.

If $q \geq k$, terminate.

Determine for all $x \in N$:

$\pi(x)$ = shortest distance from 0 to x .

$\eta(x)$ = shortest distance from x to D .

Note that P^q_{q+1} does not exist (the length of a route which does not exist is taken as infinite).

$k_q = 0$.

Step 1. - Let:

(1) $$\zeta_q(x) = \begin{cases} \text{Min}_{(y,x)\,\varepsilon\,\bar{A}_q} \{\pi(y)+a(y,x)\} & \text{for } x \in N_q \\ \pi(x) & \text{for } x \notin N_q \end{cases}$$

(2) $$E = \{x \in N \mid \zeta_q(x) + \eta(x) = \underset{y\varepsilon N}{\text{Min}}\,[\zeta_q(y) + \eta(y)] = \bar{\pi}_q\}$$

If $\bar{\pi}_q = \infty$ and $k_q + q < k$: terminate, the problem is infeasible.

Determine m by:

$$\ell(P^q_{q+m}) \leq \bar{\pi}_q < \ell(P^q_{q+m+1})$$

Let:

$$P_{q+i} = P^q_{q+i} \qquad i = 1,..,m$$

$$G_{q+m} = G_q$$

$$P^{q+m}_{q+m+i} = P^q_{q+m+i} \qquad i = 1,....,(k_q - m)$$

$$q = q + m$$

$$j = 0$$

For each $x \varepsilon E$ generate all the routes $R_{(ox)}$ from 0 to x, the last arc of which belongs to $\bar{A}_q$, and, the length of which is $\zeta_q(x)$; this is easily done by backtracking in the definition of $\zeta_q(x)$ and $\eta(x)$. Similarly generate all the chains $C_{(xD)}$ from x to D of length $\eta(x)$.

By combining a route $R_{(ox)}$ with a chain $C_{(xD)}$ and repeating the operation for all $x \varepsilon E$, all the routes from 0 to D of length $\bar{\pi}_q$ can be generated. Lets call them $\bar{P}^q_{q+1}, \ldots, \bar{P}^q_{q+n}$. We need only to generate at most $k - q$ of them.

If $n \geq k - q$ let $P_{q+j} = \bar{P}^q_{q+j}$ $j = 1,....,k - q$; terminate, solution has been obtained.

Go to Step 2.

<u>Step 2</u>. - Let $j = j + 1$

If $j > n$ go to Step 1.

$$P_{q+1} = \bar{P}^q_{q+j}$$

$$G_{q+1} = G_q \cup \bar{P}^q_{q+j}$$

If the set of arcs in $G_{q+1} - G_q$ is empty let:

$$P^{q+1}_{q+1+i} = P^q_{q+i} \qquad 1 \leq i \leq k_q$$

$$q = q + 1$$

Go to Step 2.

If the set of arcs in $G_{q+1} - G_q$ is not empty, they form a simple route C_j from 0_j to D_j $(0_j, D_j \in G_q)$ (of Lemma 5).

Let $R = (R_1,..,R_r)$, $S = (S_1,...,S_s)$, $T = (T_1,..,T_t)$ be respectively a set of routes from 0 to 0_j , from D_j to 0_j and from D_j to D , obtained by considering parts of:

- Those routes from 0 to D which contain 0_j ;
- Those routes from 0 to D which contain D_j and <u>then</u> 0_j ;
- Those routes from 0 to D which contain D_j .

Generate new routes in G_q , according to increasing lengths, by taking the union of a part of a route in R , from 0 to 0_j , route C_j and a part of a route in T from D_j to D . Insert those newly created routes in the sequence P^q_{q+i} so that the new sequence is still ordered according to increasing lengths. Generate new routes until exhaustion of possibilities or until the

$$p_q = k - q - n + j - 1$$

first elements of this new sequence have been created.

If $S = \emptyset$ go to Step 4.

Let $u = 1$. We will call route of order u a route which contains $(u+1)$ times route C_j (and u times an element of S). For instance a route of order 1 is the union of a route in R , route

C_j , a route in S , route C_j , a route in T .

Go to Step 3.

Step 3. - Generate routes of order u according to increasing lengths and their elements in the sequence just generated in such a position that the sequence is still ordered according to increasing lengths either until exhaustion of possibilities or until the p_q first elements of the sequence have been generated.

If the first route of order u is among the p_q first elements let $u = u + 1$ and go to Step 3.

If not, go to Step 4.

Step 4. - Let $q = q + 1$

Denote by P^q_{q+i} the elements of the sequence just generated and by k_q their number.

Go to Step 2.

3. Justification Of The Algorithm

Let

$$\tilde{A} = \{(x,y) \in A \mid \pi(y) - \pi(x) = a(x,y)\}$$

$$\tilde{G} = (N,\tilde{A})$$

Lemma 1. - A necessary and sufficient condition for a cycle to be of 0 length (0-cycle) is that it be contained in $\tilde{G}$.

Proof:

(i) If a cycle $[x_o,x_1,\ldots,x_{n-1},x_n = x_o]$ belongs to $\tilde{G}$, it is a 0 cycle.

$$\begin{array}{rcl} \pi(x_1) - \pi(x_o) & = & a(x_o,x_1) \\ \vdots & & \vdots \\ \pi(x_o) - \pi(x_{n-1}) & = & a(x_{n-1},x_o) \\ \hline 0 & = & \sum_{i=1}^{n} a(x_{i-1},x_i) \end{array}$$

(ii) Let C be a 0 cycle. For all the arcs $(x,y) \varepsilon C$ we have by definition of π :

$$\pi(y) - \pi(x) \leq a(x,y)$$

If for one arc we had strict inequality by adding these relations for all arcs of C , we would obtain $0 < \ell(C)$ which contradicts the assumption.

q.e.d.

Lemma 2. - All routes in $\tilde{G}$ from 0 to a given node z have the same length: $\pi(z)$. Any route not completely included in $\tilde{G}$ has a larger length.

Proof:

Obvious, since along any route of $\tilde{G}$

$$\pi(y) - \pi(x) = a(x,y)$$

and for a route not completely included in $\tilde{G}$, we would have on at least one arc: $\pi(y) - \pi(x) < a(x,y)$.

q.e.d.

Lemma 3. - A necessary and sufficient condition for n to be finite at the end of Step 1 is that P^q_{q+j} do not contain a 0-cycle.

Proof:

Suppose $\bar{P}^q_{q+j}$ contains a 0-cycle, then the routes obtained by the union of $\bar{P}^q_{q+j}$ and i times this 0-cycle are of same length $\bar{\pi}_q$ for any positive integer i .

If there is an infinite number of routes of length $\bar{\pi}_q$, there is an infinite subset of these routes the elements of which differ from one another by the addition of a cycle a certain number of times since the number of chains from 0

to D is finite. The length of this cycle is thus 0 .

q.e.d.

Lemma 4. - $\bar{\pi}_q$, as defined in (2) , is the shortest length of all routes not completely included in G_q .

Proof:

It is sufficient to show that for each route P^* which contains at least an arc (x,y) in $\bar{A}_q$, $\ell(P^*) \geq \bar{\pi}_q$.

From the definition of π and η we have:

$$\ell(P^*) \geq \pi(x) + a(x,y) + \eta(y) \tag{3}$$

But from (1) :

$$\pi(x) + a(x,y) \geq \zeta_q(y)$$

and the property of shortest distances that:

$$\pi(y) \leq \pi(x) + a(x,y)$$

Then, from (2) :

$$\zeta_q(y) + \eta(y) \geq \bar{\pi}_q$$

which together with (3) implies:

$$\ell(P^*) \geq \bar{\pi}_q \quad .$$

q.e.d.

Lemma 5. - In Step 2 the set of arcs in $G_{q+1} - G_q$ is either empty or forms a simple route C_j from O_j to D_j $(O_j, D_j \in G_q)$. This route is either a chain or a simple cycle in which case $D_j = O_j$.

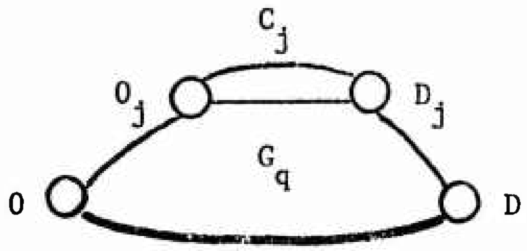


Figure 1

Proof:

If the set of arcs in $G_{q+1} - G_q$ is not empty, $\bar{P}^q_{q+j}$ includes successive sequences of arcs of $\bar{A}_q$; let C_j be the last of these sequences when $\bar{P}^q_{q+j}$ is described from 0 to D . C_j forms a route from 0_j (ε G_q) to D_j (ε G_q) since C_j is a subset of $\bar{P}^q_{q+j}$.

The part of $\bar{P}^q_{q+j}$ from 0 to 0_j is included in $\tilde{G}$ for if it were not, it would be possible, from Lemma 2, to construct another route shorter than $\bar{\pi}_q$ and which would have the arcs of C_j outside of G_q thus contradicting Lemma 4 and similarily for D_j to D , proving $\exists$ a single path C_j .

Suppose that a proper subset of C_j forms a cycle. If this cycle was a 0-cycle from Lemma 3, n would be infinite and the algorithm would have been terminated at the end of Step 1.

If this cycle had a positive length, it would be possible by deleting it from $\bar{P}^q_j$ to construct a route not completely included in $\bar{G}_q$ and of length $< \bar{\pi}_q$, thus contradicting Lemma 4 again. Thus if there is a cycle, it can only be simple.

q.e.d.

The following lemma is not crucial for the proof of the theorem, however, it helps understanding of the algorithm.

Lemma 6. - $\bar{\pi}_q > \bar{\pi}_{q-1}$

Proof:

It is impossible that $\bar{\pi}_q < \bar{\pi}_{q-1}$ since $\bar{P}^q_{q+j}$ is not among the q shortest

routes.

Suppose $\bar{\pi}_q = \bar{\pi}_{q-1}$. Then $\exists\,(x,y) \in \bar{A}_q \cap \bar{P}^q_{q+1}$ for which $\zeta_q(y) + \eta(y) = \bar{\pi}_q = \bar{\pi}_{q-1}$ but then $\bar{P}^q_{q+1}$ would have been discovered at a previous passage and (x,y) would have been added to $\bar{A}_q$, thus a contradiction.

q.e.d.

Theorem

The algorithm leads to a solution in a finite number of steps, or shows that no solution exists.

Proof:

In Steps 2 and 3 the sequence $P^q_{q+1}, \ldots, P^q_{q+k_q}$ is generated so that this sequence is at hand at beginning of Step 1. Then P_{q+1} is either the shortest route completely included in G_q , P^q_{q+1} , or the shortest route containing some arc in $\bar{A}_q$, $\bar{P}^q_{q+1}$ (Lemma 4).

If the number of routes in G_q , $(k_q + q)$, is less then k and if there does not exist routes outside those in G_q $(\bar{\pi}_q = \infty)$ the problem is infeasible.

Each time Step 1 is described, some arcs are added to G_q so that we can only go a finite number of times through Step 1.

We cannot cycle indefinitely in Step 3 since each time we go through Step 3, at least, a new route is added to the updated sequence and we do not want more than p_q routes.

Now we can go only a finite number of times through Step 2 since at each passage the index j is increased by one unit and when $j > n$ we go to Step 1.

q.e.d.

4. Variant Of The Preceding Algorithm To Find The k Shortest Chains

The basic idea of this variant is to perform the algorithm of Section 2 and to keep the routes thus generated only if they are chains. Two main differences

with the initial algorithm justify a rewriting of the algorithm:

- When a 0-cycle is met, no new chain will ever be constructed.
- Routes of order $u \geq 1$ will not give rise to any chain and Step 3 can be skipped.

Step 0'. - Find all the shortest chains from 0 to D : $P'_1, .., P'_q$.

If $q \geq k$, terminate.

Determine for all $x \in N$:

$\pi(x)$ = shortest distance from 0 to x .

$\eta(x)$ = shortest distance from x to D .

$k'_q = 0$.

Step 1'. - Define:

$$(1) \qquad \zeta_q(x) = \begin{cases} \operatorname{Min}_{(y,x) \in \bar{A}_q} \{\pi(y) + a(y,x)\} & \text{for } x \in N_q \\ \pi(x) & \text{for } x \notin N_q \end{cases}$$

$$(2) \qquad E = \{x \in N \mid \zeta_q(x) + \eta(x) = \operatorname{Min}_{y \in N} [\zeta_q(y)+\eta(y)] = \bar{\pi}_q\}$$

If $\bar{\pi}_q = \infty$ and $k'_q + q < k$: terminate, the problem is infeasible.

Determine m by:

$$\ell(P'^q_{q+m}) \leq \bar{\pi}_q < \ell(P'^q_{q+m+1})$$

Let:

$P'_{q+i} = P'^q_{q+i}$ $\qquad i = 1,....,m$

$G'_{q+m} = G'_q$

$P'^{q+m}_{q+m+i} = P'^q_{q+m+i}$ $\qquad i = 1,..,(k'_q - m)$

$q = q + m$

$j = 0$

For each $x \in E$ generate the routes $R_{(ox)}$ from 0 to x, the last arc of which belongs to $\bar{A}_q$ and the length of which is $\zeta_q(x)$; this is easily done by backtracking in the definition of $\zeta_q(x)$ and $\pi(x)$. Similarly generate all the chains $C_{(xD)}$ from x to D of length $\eta(x)$.

By combining a route $R_{(ox)}$ which a chain $C_{(xD)}$ in such a way that <u>no newly constructed route from 0 to D contains a cycle counted more than once</u>, a finite number n of routes from 0 to D of length $\bar{\pi}_q$, $\bar{P}^q_{q+1}, \ldots, \bar{P}^q_{q+n}$ are generated. Among those are all the chains of length $\bar{\pi}_q$ and having a nonempty intersection with $\bar{A}_q$.

Go to Step 2'.

<u>Step 2'</u>. - $j = j + 1$

If $j > n$ go to Step 1'.

If $\bar{P}^q_{q+j}$ is a chain, let $P'_{q+1} = \bar{P}^q_{q+j}$.

Let

$$q' = \begin{cases} q+1 & \text{if } \bar{P}^q_{q+j} \text{ is a chain} \\ q & \text{otherwise} \end{cases}$$

$$G'_{q'} = G'_q \cup \bar{P}^q_{q+j}$$

If the set of arcs in $G'_{q'} - G'_q$ is empty, let

$$k'_{q'} = k'_q$$

$$P'^{q'}_{q'+i} = P'^q_{q+i} \qquad 1 \leq i \leq k'_q$$

$$q = q'$$

Go to Step 2'.

If the set of arcs in $G'_{q'} - G'_q$ is not empty, it forms a simple route C_j from 0_j to D_j. Let C'_j be the subset of C_j which is a chain from 0_j to D_j.

Let

$$R' = (R'_1 \,..\, R'_r) \;,\; T' = (T'_1 \,..\, T'_s)$$

be respectively a set of chains from 0 to 0_j, from D_j to D obtained by taking parts of:

- Those chains from 0 to D which contain 0_j.
- Those chains from 0 to D which contain D_j.

Generate new routes in $G'_{q'}$, according to increasing lengths, by routes taking union of a chain in R', chain C'_j and a chain in T'. Keep only those routes which are chains and insert them in the sequence P'^{q}_{q+i} so that the new sequence be still ordered according to increasing lengths. Generate new chains until exhaustion of possibilities or until the $k - q + j - 1$ first elements of the new sequence have been generated.

Let

$$q = q'$$

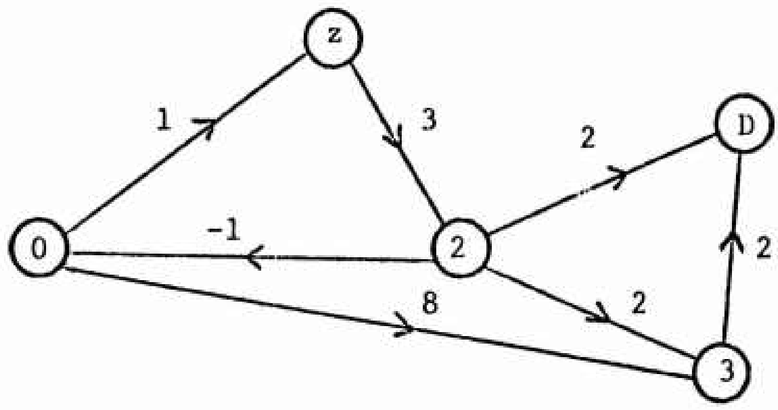
Denote by P'^{q}_{q+i} the elements of the sequence just generated and by k'_q their number.

Go to Step 2'.

This algorithm, as well as algorithm of Section 2, can be slightly modified in order to produce, respectively all the chains (routes) which are shorter than a given length L (or within $x\%$ of the length of the shortest chain (route)).

5. Examples

Suppose we want to find the four shortest routes and the four shortest chains in the following graph:



a) Four Shortest Routes

1) Step 0. - The shortest route is unique $P_1 = [0, 1, 2, D]$ $A_1 = \{(01), (12), (2D)\}$ $\ell(P_1) = 6$

2) Step 1. -

Nodes	$\pi(x)$	$\eta(x)$	$\zeta_1(x)$	$\eta(x)+\zeta_1(x)$
0	0	6	3	9
1	1	5	∞	∞
2	4	2	∞	∞
3	6	2	6	8
D	6	0	8	8

$E = \{3, D\}$

$\bar{\pi}_1 = 8$

P_2' does not exist

$\bar{P}_2' = [0, 1, 2, 3, D]$

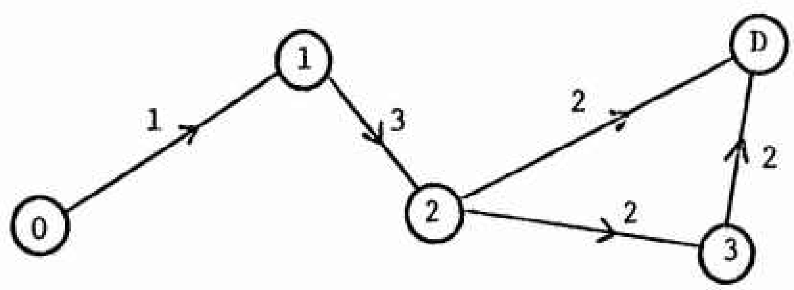
$n = 1$

3) <u>Step 2</u>. -

$$j = 1$$

$$P_2 = \bar{P}'_2$$

$$G_2 = G_1 \cup \bar{P}'_2 \quad \text{i.e. :}$$



There is no other route in G_2 than P_1 and P_2, i.e. P_3^2 does not exist. $k_2 = 0$.

Go to Step 2.

4) <u>Step 2</u>. - $j = 2 > n$: go to Step 1.

5) <u>Step 1</u>. -

Nodes	$\eta(x)$	$\zeta_2(x)$	$\eta(x)+\zeta_2(x)$
0	6	3	9
1	5	∞	∞
2	2	∞	∞
3	2	8	10
D	0	∞	∞

$E = \{0\}$

$\bar{\pi}_2 = 9$

Since P_3^2 does not exist $m = 0$

$$\bar{P}_3^2 = [0, 1, 2, 0, 1, 2, D]$$

$$n = 1$$

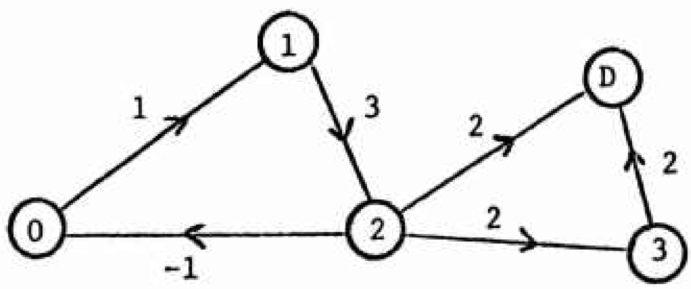
Go to Step 2.

6) <u>Step 2</u>. -

$$j = 1$$

$$P_3 = \bar{P}_3^2$$

$$G_3 = G_2 \cup \bar{P}_3^2 \qquad \text{i.e.}$$



$$C_3 = (2, 0)$$

$$P_4^3 = [0\text{-}1\text{-}2\text{-}0\text{-}1\text{-}2\text{-}3\text{-}D]$$

$$\ell(P_4^3) = 11$$

If we go to Step 3 and generate the shortest route of order 1, we find

$$\tilde{P}_4^3 = [0\text{-}1\text{-}2\text{-}u\text{-}1\text{-}2\text{-}0\text{-}1\text{-}2\text{-}D]$$

$$\ell(\tilde{P}_4^3) = 12$$

and we insert it in the sequence after P_4^3 .

Go to Step 2.

7) <u>Step 2</u>. - $j = 2 > n - 1$, go to Step 1.

8) <u>Step 1</u>. -

Nodes	$\eta(x)$	$\zeta_3(x)$	$\eta(x)+\zeta_3(x)$
0	6	∞	∞
1	5	∞	∞
2	2	∞	∞
3	2	8	10
D	0	∞	∞

$E = \{3\}$

$\bar{\pi}_3 = 11$

and

$$P_4 = \bar{P}_4^3 = [0\text{-}3\text{-}D]$$

terminate.

b) Four Shortest Chains

Up to stage (6) the two algorithms give the same result i.e.

$$P_1' = [0,\ 1,\ 2,\ D]$$

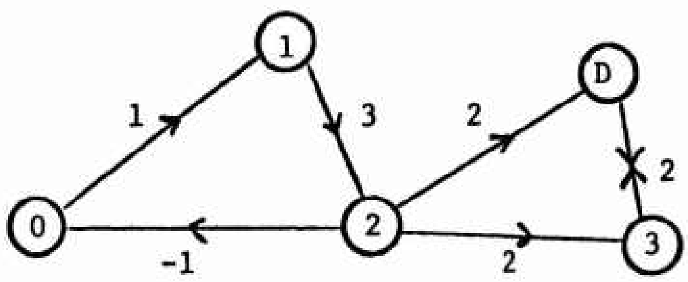
$$P_2' = [0,\ 1,\ 2,\ 3,\ D]$$

6') Step 2'. -

$$j = 1$$

$$q' = q = 2$$

$$G_2' = G_2' \cup \bar{P}_3^2$$



No new chain in G_2' ; $P_3'^2$ does not exist. $k_2' = 0$.

Go to Step 2'.

7') <u>Step 2'</u>. - $j = 2 > n = 1$ go to Step 1'.

8') <u>Step 1'</u>. -

Nodes	$\eta(x)$	$\zeta_2(x)$	$\eta(x)+\zeta_2(x)$
0	6	∞	∞
1	5	∞	∞
2	2	∞	∞
3	2	8	10
D	0	∞	∞

$E = \{3\}$

$\bar{\pi}_3 = 10$

$P_3'^{2}$ does not exist so $m = 0$

$\bar{P}_3^2 = [0, 3, D]$

$n = 1$

Go to Step 2'.

9') <u>Step 2'</u>. - $j = 1$

$\bar{P}_3^2$ is a chain: $P_3' = \bar{P}_3^2 = [0, 3, D]$

$$q' = 3$$

$$G_3' = G_2' \cup \bar{P}_3^2 = G$$

There is no chain outside P_1' P_2' P_3' in G_3' ; i.e., $P_4'^{3}$ does not exist. $k_3' = 0$.

Go to Step 1'.

10') <u>Step 1'</u>. -

Nodes	$\eta(x)$	$\zeta_3(x)$	$\eta(x)+\zeta_3(x)$
0	6	∞	∞
1	5	∞	∞
2	2	∞	∞
3	2	∞	∞
D	0	∞	∞

$\bar{\pi}_3 = \infty$

$$k_3' + 3 = 0 + 3 < 4$$

Problem is infeasible.

REFERENCES

[1] Bellman, R., and R. Kalaba, "On k^{th} Best Policies," Journal of SIAM, Vol. 8, pp 582-588, (1960).

[2] Berge, C., "Theorie des Graphes et Applications," Dunod, (1962).

[3] Clarke, S., Krikorian, A., and J. Rausen, "Computing the N Best Loopless Paths in a Network," Journal of SIAM, Vol. 11, No. 4, pp 1096-1102.

[4] Dantzig, G. B., "The Shortest Route Problem," Operations Research Journal, Vol. 5, pp 270-273, (1951).

[5] Ford, L., and R. Fulkerson, "Flows in Networks," Princeton University Press, pp 130-134.

[6] Hoffman, W., and R. Parley, "A Method for the Solution of the N^{th} Best Path Problem," Journal of Association Computing Machinery, Vol. 6, pp 506-514.

[7] Pollack, M., "Solution of the k^{th} Best Route Through a Network," A Review, Journal of Mathematical Analysis and Application, Vol. 3, pp 547-559, (1961).

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of California, Berkeley	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

THE k SHORTEST ROUTES AND THE k SHORTEST CHAINS IN A GRAPH

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Research Report

5. AUTHOR(S) (Last name, first name, initial)

Sakarovitch, Michel

6. REPORT DATE October 1966	7a. TOTAL NO. OF PAGES 24	7b. NO. OF REFS 7
8a. CONTRACT OR GRANT NO. Nonr-222(83) b. PROJECT NO. NR 047 033 c. RR 003 07 01 d.	9a. ORIGINATOR'S REPORT NUMBER(S) ORC 66-32 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	

10. AVAILABILITY/LIMITATION NOTICES

Available without limitation on dissemination.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Mathematical Science Division

13. ABSTRACT

DD FORM 1 JAN 64 1473

Unclassified
Security Classification

Unclassified
Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
k Shortest Routes						
k Shortest Chains						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
______________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
______________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
______________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Unclassified
Security Classification